This is a study guide for Exam 4. You are expected to understand and be able to answer mathematical questions on the following topics.

Chapter 32

Self-Induction and Induction

While a battery creates an emf and current, and changing magnetic field creates an *induced* emf and *induced* current. The current in a loop creates a magnetic field that passes through the loop. When the current first starts, that magnetic field is changing as the current goes from zero to a value. Albeit small, this changing magnetic field creates an induced current in the opposite direction of the emf current, and it is called a *back emf*. This effect is called **self-induction** which creates a **self-induced emf**.

An induced emf is the negative of a changing magnetic field. Similarly, a self-induced emf would be found by

$$\mathcal{E} = -L\frac{dI}{dt}$$

where L is a constant called the **inductance** of the loop. It depends on the physical characteristics of the loop, so it is a constant for that circuit. If we compare this to a solenoid, which has an induced emf of

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

$\Phi_B = \mu_o nI$, so

$$\mathcal{E} = -N\frac{d(\mu_o nI)}{dt} = -N\mu_o n\frac{dI}{dt} = -\frac{N\Phi_B}{I}\frac{dI}{dt}$$

we can say the inductance of an N-turn coil is

$$L = \frac{N\Phi_B}{I}$$

We can also say

$$L = -\frac{\mathcal{E}}{dI/dt}$$

While resistance is the emf over current, inductance is the emf over the change in current. The units of inductance is a henry (H) where $1H = 1V \cdot s/A$.

RL Circuits

An RL circuit contains a battery, a coil (solenoid) and a resistor. The coil acts as an **inductor**, resisting changes in current. If a voltage is applied or stopped, the inductor won't allow the current to go to max or min immediately.

Increasing Current

If the current in an RL circuit is increasing, the current in the circuit can be found by

$$I = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau})$$

where $\tau = L/R$ is the time constant of an RL circuit.

Decreasing Current

If the battery is removed, and only a resistor and coil are present, the current will not simply stop because the battery is no longer present. The inductor resists the change in current, so the current will decay exponentially:

$$I = \frac{\mathcal{E}}{R}e^{-t/\tau} = I_{max}e^{-t/\tau}$$

I_{max} depends on the current through the inductor when the current begins to decrease.

Energy in a Magnetic Field of the Inductor

The current in the RL circuit doesn't simply stop when the battery was removed, and the energy that keeps the current going is stored in the magnetic field of the inductor:

$$U = \frac{1}{2}LI^2$$

where L is the inductance of the inductor and I is the current.

Mutual Inductance

A magnetic field may exist in a circuit due to an inductance in another nearby circuit. This results in **mutual inductance**, where the inductance in a system depends on two circuits. If two coils are close to each other, the magnetic flux created by coil 1 passes through coil 2 would be Φ_{12}. Thus, the mutual inductance would be

$$M_{12} = \frac{N_2\Phi_{12}}{I_1}$$

The mutual inductance depends on the physical characteristics and geometry of the two circuits. The emf in coil 2 induced by coil 1 would then be

$$\mathcal{E}_2 = -M_{12}\frac{dI_1}{dt}$$

However, coil 2 may also have a current, in which case

$$\mathcal{E}_1 = -M_{12}\frac{dI_2}{dt}$$

The mutual inductance is the same in both cases, $M_{12} = M_{21} = M$.

The LC Circuit

An LC circuit is made up of an inductor and a capacitor without a power supply. Assume the capacitor is initially charged. If the capacitor is allowed to discharge across the inductor, the inductor will resist the change in current. As the capacitor continues to discharge, energy is stored in the magnetic field of the inductor. When the capacitor has discharged, the inductor resists a change in current, and keeps it going. It will then charge the capacitor, but with an opposite polarity as before.

It is analogous to a spring with a mass, where the energy is transferred back and forth between kinetic and potential. The capacitor is like the potential energy of the spring. When the mass is released, the momentum resists stopping, just like the inductor resists the change in current. The energy will continue to resonate between the capacitor and the inductor. The energy stored in the charged capacitor is $\frac{Q^2}{2C}$ while the energy stored in the magnetic field of the inductor is $\frac{1}{2}LI^2$. The total energy in the system is then

$$U = U_C + U_L = \frac{Q^2}{2C} + \frac{1}{2}LI^2$$

In order to conserve energy, the maximum energy in the capacitor would have to be the same as the maximum energy in the inductor. Thus, we can say

$$\frac{Q_{max}^2}{2C} = \frac{1}{2}LI_{max}^2$$

Assuming that at time $t = 0$, all of the energy in the system is stored as charge on the capacitor and there is no current, the charge on the capacitor as a function of time can be found as

$$Q(t) = Q_{max}\cos(\omega t)$$

where Q_{max} is the maximum charge that will be on the capacitor. The angular frequency ω is the *natural frequency* at which the current in the system and the charge on the capacitor will oscillate, found by

$$\omega = \frac{1}{\sqrt{LC}}$$

The current in the circuit can be found by

$$I = \frac{dQ}{dt} = -\omega Q_{max}\sin(\omega t + \phi) = -I_{max}\sin(\omega t + \phi)$$

Because we assume the capacitor is initially charged with no current in the circuit, we can say that at time $t = 0$,

$$0 = -\omega Q_{max}\sin\phi$$

which means $\phi = 0$ so the current in the circuit will be

$$I(t) = -I_{max}\sin(\omega t)$$

The RLC Circuit (without a power supply)

The LC circuit is idealized, whereas real circuits lose energy due to radiation and resistance. This loss in energy will result in a dampening of the current in the circuit and charge stored on the capacitor that will decay exponentially. For a given resistance in the circuit, we need to add a dampening component of $e^{-Rt/2L}$. Again assuming that at time $t = 0$, there is no current in the circuit and the capacitor is charged, the charge on the capacitor as a function of time can be found by

$$Q(t) = Q_{max}e^{-\frac{Rt}{2L}}\cos(\omega_d t)$$

The natural frequency will also need to take into account the resistance, and this damped frequency ω_d can be found by

$$\omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

Chapter 33

AC Sources

An AC circuit has a power supply with voltage

$$\Delta v = \Delta V_{max}\sin\omega t$$

where ΔV_{max} is the maximum voltage. Because the voltage is sinusoidal, it is positive half the time and negative the other half. The angular frequency of the AC voltage is

$$\omega = 2\pi f = \frac{2\pi}{T}$$

where f is the frequency and T is the period. Households have $f = 60\ Hz$ which corresponds to $\omega = 377\ rad/s$.

Resistors in an AC Circuit

If an AC source is in a circuit with a resistor, the current through the resistor as a function of time would be

$$i_R = I_{max}\sin\omega t$$

where

$$I_{max} = \frac{\Delta V_{max}}{R}$$

The instantaneous voltage across the resistor is

$$\Delta v_R = I_{max}R\sin\omega t$$

Note: The direction in which we call the voltage positive or negative is arbitrary. Because the current and voltage in this system vary identically in time, they are said to be *in phase*. The average current is found by the **rms current**:

$$I_{rms} = \frac{I_{max}}{\sqrt{2}} = 0.707 I_{max}$$

The rms current can then be used to find the power being delivered to the resistor:

$$P_{avg} = I_{rms}^2 R$$

Because energy is being dissipated only by the resistor, this is the power going to the circuit. The **rms voltage** would be

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = 0.707\Delta V_{max}$$

Inductors in an AC Circuit

However, if an inductor is in the circuit, the current and voltage will not be in phase. To analyze the phases in AC circuits, we use a **phasor diagram**, where a **phasor** is the represented by a vector whose length is proportional to the maximum value of the variable it is representing. We then have the vector rotate counter-clockwise around the origin, and the vertical axis represents the instantaneous value for the variable.

If an inductor is in the circuit with an AC power supply, it will resist the changes in current. The instantaneous current through the inductor will be

$$i_L = \frac{\Delta V_{max}}{X_L} \sin\left(\omega t - \frac{\pi}{2}\right)$$

where X_L is the **inductive reactance**, found by

$$X_L = \omega L$$

The current is then out of phase with the voltage by $\pi/2$, with the current lagging behind the voltage. The maximum current will be

$$I_{max} = \frac{\Delta V_{max}}{X_L}$$

so the maximum voltage across the inductor can be found by

$$\Delta V_{max} = I_{max} X_L$$

Capacitors in an AC Circuit

The instantaneous current in an AC circuit with a capacitor would be

$$i_C = \frac{\Delta V_{max}}{X_C} \sin(\omega t + \pi/2)$$

where X_C is the **capacitive reactance**, found by

$$X_C = \frac{1}{\omega C}$$

The current is then out of phase with the voltage by $\pi/2$, with the current leading the voltage. The maximum current in the system would again be

$$I_{max} = \frac{\Delta V_{max}}{X_C}$$

Thus, like before, the maximum voltage would be

$$\Delta V_{max} = I_{max} X_C$$

The RLC Series Circuit

An RLC circuit has a capacitor, inductor, and capacitor in a circuit with an AC power source. The instantaneous power supplied is

$$\Delta v = \Delta V_{max} \sin \omega t$$

and the instantaneous current is then

$$i = I_{max} \sin(\omega t - \phi)$$

where ϕ is a **phase angle** between the current and applied voltage. Because the circuit elements are in series, the instantaneous current is the same everywhere (amplitude and phase). However, the instantaneous voltage will be different across the different elements (amplitude and phase). The voltage across the resistor will be in phase with the current, while the voltage across the inductor will lead by 90° and the voltage across the capacitor will lag by 90°. The instantaneous voltages will then be

$$\Delta v_R = I_{max} R \sin \omega t = \Delta V_R \sin \omega t$$

$$\Delta v_L = I_{max} X_L \sin\left(\omega t + \frac{\pi}{2}\right) = \Delta V_L \cos \omega t$$

$$\Delta v_C = I_{max} X_C \sin\left(\omega t - \frac{\pi}{2}\right) = -\Delta V_C \cos \omega t$$

When adding these voltages together, it is easiest to sum the rotating vectors of the phasors. This leads to the impedance of the circuit, which is a combination of the resistance and reactance. The impedance Z can by found by

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

The maximum current in the system is found by

$$I_{max} = \frac{\Delta V_{max}}{Z}$$

and the phase angle is found by

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

If the circuit is more inductive than capacitive ($X_L > X_C$, which occurs at high frequencies), then the phase angle is positive and the current lags behind the applied voltage. If the circuit is more capacitive than inductive ($X_L < X_C$), then phase angle is negative and the current leads the applied voltage. When $X_L = X_C$, then the phase angle is zero and the circuit is purely resistive. When the phase angle is zero, the current is maximized so this is when maximum power is going to the resistor. This is done when the power supply is oscillating at the *resonance frequency*

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Transformers

An AC transformer uses two coils that are wound around the same iron core. The iron core directs the magnetic flux created by the primary coil and directs it through the secondary coil. The primary coil is attached to an input alternating voltage power, which creates an oscillating magnetic flux. This oscillating magnetic flux will induce an alternating voltage in the secondary coil. The voltage across the primary coil is found by

$$\Delta v_1 = -N_1 \frac{d\Phi_B}{dt}$$

where N_1 is the number of turns in the primary coil. If all of the magnetic flux is directed through the secondary coil, then

$$\Delta v_2 = -N_2 \frac{d\Phi_B}{dt}$$

where N_2 is the number of turns in the secondary coil. We can thus find the voltage created in the secondary coil by

$$\Delta v_2 = \frac{N_2}{N_1} \Delta v_1$$

If $N_2 > N_1$, then this is a *step-up transformer*, resulting in $\Delta v_2 > \Delta v_1$. If $N_2 < N_1$, this is a *step-down transformer*, resulting in $\Delta v_2 < \Delta v_1$. There will be no power loss through an ideal transformer, which means

$$I_1 \Delta V_1 = I_2 \Delta V_2$$

where I_1 and ΔV_1 are the rms current and voltage in the primary circuit, and I_2 and ΔV_2 are the rms current and voltage in the secondary circuit.